Case 1) $\alpha = 0$, $\beta = 0$, $\gamma = 1$ (Hiamang and Mickens[1])
Case 2) $\alpha = 0$, $\beta = -1$, $\gamma = 1$ (Gottlieb[6])
Case 3) $\alpha = 1$, $\beta = -2.2518$, $\gamma = 2.54328$ (Nageswara Rao[5])

The angular frequencies ω for the specified values of the positive amplitude x_+ are presented in Table 1.

Conclusions

The harmonic balance method with lower-order harmonics yields good results that are comparable with those obtained by exact integration. The frequency–amplitude relation obtained for the Duffing-type equation using the method of harmonic balance with lower-order harmonics will be useful for all practical purposes in understanding the nonlinear free-vibration characteristics of composite structures.

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Singularities in Polynomial Representations of Transverse Shear in Finite Elements

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I. Introduction

POLYNOMIALS are frequently used to represent transverse shear when modeling composite shells with two-dimensional finite elements. Layerwise theories have been developed that capture the continuous transverse shear stresses and displacements through the thickness of the laminate, but such schemes do not permit the use of complete cubic polynomials in representing the displacement functions: the number of unknown coefficients exceeds the number of conditions imposed on the laminate. Therefore, choices must be made regarding the best representation of an incomplete cubic, and only a few choices make physical sense. Of these, it is shown that (at least) one choice for these polynomials exists that can produce ill-behaved (even singular) displacement functions.

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II. Theory

Use of higher-order shear theories (HSTs) to represent transverse shear behavior usually involves representing the in-plane displacements u_i, $i \in \{1, 2\}$, as a function of the laminate thickness coordinate z, by polynomials having unknown coefficients. Moreover, one may choose to represent displacements in each ply of the laminate,[1] such that the displacements within the kth ply of a laminate have the form

$$u_1^{(k)}(z) = \left(a_{14}^{(k)} + b_{14}^{(k)} z + c_{14}^{(k)} z^2 + d_{14}^{(k)} z^3\right)\gamma_4 + \left(a_{15}^{(k)} + b_{15}^{(k)} z + c_{15}^{(k)} z^2 + d_{15}^{(k)} z^3\right)\gamma_5 \tag{1a}$$

$$u_2^{(k)}(z) = \left(a_{24}^{(k)} + b_{24}^{(k)} z + c_{24}^{(k)} z^2 + d_{24}^{(k)} z^3\right)\gamma_4 + \left(a_{25}^{(k)} + b_{25}^{(k)} z + c_{25}^{(k)} z^2 + d_{25}^{(k)} z^3\right)\gamma_5 \tag{1b}$$

where the angles γ_4 and γ_5 represent local shear rotation in the 2–3 (y–z) and 1–3 (x–z) planes, respectively. To find these displacements, one must find $16N$ unknown coefficients (where N is the total number of plies in the laminate), $a_{mn}^{(k)}$, $b_{mn}^{(k)}$, $c_{mn}^{(k)}$, and $d_{mn}^{(k)}$.

Because continuity of interlaminar stresses and displacements do not provide all of the $16N$ equations necessary, the displacement functions must be simplified by omitting coefficients.

The theory presented by Pai and Palazotto[2] incorporates a local and layerwise displacement theory using the Jaumann (or Biot–Cauchy–Jaumann) strains B_{mn} and stresses J_{mn}. Jaumann measures are equivalent to local engineering measures and, hence, are suitable for analyzing large deformation and small strains.

By enforcing shear stress continuity and in-plane displacement continuity at each ply interface, as well as assuming a shear strain-free condition on the exterior surfaces, the resulting $4N$ algebraic equations can be stated as

$$B_{23}^{(1)}(x, y, z_1) = 0, \qquad B_{13}^{(1)}(x, y, z_1) = 0 \tag{2a}$$

$$u_1^{(k)}(x, y, z_{k+1}) - u_1^{(k+1)}(x, y, z_{k+1}) = 0 \quad \text{for} \quad k = 1, \ldots, N-1 \tag{2b}$$

$$u_2^{(k)}(x, y, z_{k+1}) - u_2^{(k+1)}(x, y, z_{k+1}) = 0 \quad \text{for} \quad k = 1, \ldots, N-1 \tag{2c}$$

$$J_{23}^{(k)}(x, y, z_{k+1}) - J_{23}^{(k+1)}(x, y, z_{k+1}) = 0 \quad \text{for} \quad k = 1, \ldots, N-1 \tag{2d}$$

$$J_{13}^{(k)}(x, y, z_{k+1}) - J_{13}^{(k+1)}(x, y, z_{k+1}) = 0 \quad \text{for} \quad k = 1, \ldots, N-1 \tag{2e}$$

$$B_{23}^{(N)}(x, y, z_{N+1}) = 0, \qquad B_{13}^{(N)}(x, y, z_{N+1}) = 0 \tag{2f}$$

These $4N$ equations become $8N$ equations when we insist that they be satisfied for γ_4 and γ_5 independently, but this is still short of the requisite $16N$ needed by Eqs. (1). Pai and Palazotto[2] suggested the following incomplete cubic forms for the displacement functions:

$$\begin{aligned} g_{14}^{(k)} &\equiv c_{14}^{(k)} z^2 + d_{14}^{(k)} z^3, & g_{15}^{(k)} &\equiv z + c_{15}^{(k)} z^2 + d_{15}^{(k)} z^3 \\ g_{24}^{(k)} &\equiv z + c_{24}^{(k)} z^2 + d_{24}^{(k)} z^3, & g_{25}^{(k)} &\equiv c_{25}^{(k)} z^2 + d_{25}^{(k)} z^3 \end{aligned} \tag{3}$$

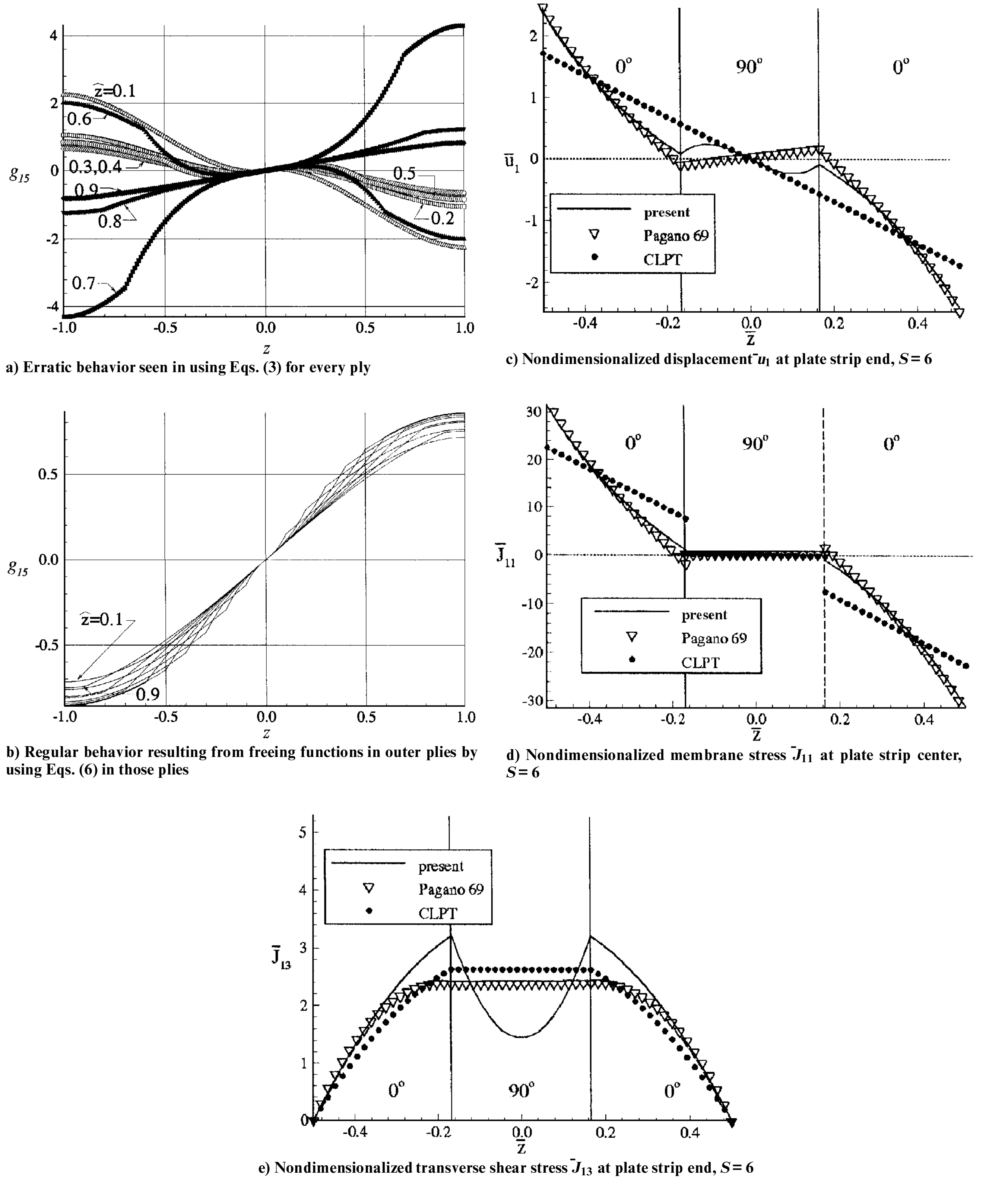
While using the functions of Eqs. (3) in the context of studying the elasticity solutions of Pagano,[3] it was found that certain choices of shear moduli and ply thicknesses of a [0/90/0] laminated plate strip could induce artificially high shear stiffness.

Consider a [0/90/0] three-ply laminate with outer plies of identical thickness and the reference surface at the middle of the center ply (the thickness of which may vary). If the plate is considered to be infinitely long in the y direction (a plane strain case), displacements are given (neglecting any thickness changes) by

$$u_1^{(k)} = g_{15}^{(k)}(z)\,\gamma_5(x, y), \qquad u_2^{(k)} = 0, \qquad u_3^{(k)} = 0 \tag{4}$$

Applying the aforementioned laminae boundary conditions leads to a 6×6 system of equations in $u_i^{(k)}$. The determinant of the coefficient matrix is a function of the material properties (particularly the ratio of the transverse shear moduli, $\tilde{G} = G_{13}/G_{23}$) and the laminate thickness, and certain choices of these values make the matrix singular.

Subsequent to the work of Pai and Palazotto,[2] Pai,[1] although not mentioning the singularity problem just discussed, suggested an alternate choice of warping functions: one set of functions, of the form of Eqs. (3), to be used for the ply containing the



a) Erratic behavior seen in using Eqs. (3) for every ply

b) Regular behavior resulting from freeing functions in outer plies by using Eqs. (6) in those plies

c) Nondimensionalized displacement $\bar{u}_1$ at plate strip end, $S = 6$

d) Nondimensionalized membrane stress $\bar{J}_{11}$ at plate strip center, $S = 6$

e) Nondimensionalized transverse shear stress $\bar{J}_{13}$ at plate strip end, $S = 6$

Fig. 1 Displacement (shear warping) functions for various values of center ply thickness $\hat{z}$.

reference surface, and another set of functions to be used for all other plies:

$$g_{14}^{(k)} \equiv c_{14}^{(k)} z^2 + d_{14}^{(k)} z^3, \qquad g_{15}^{(k)} \equiv a_{15}^{(k)} + z + d_{15}^{(k)} z^3$$
$$g_{24}^{(k)} \equiv a_{24}^{(k)} + z + d_{24}^{(k)} z^3, \qquad g_{25}^{(k)} \equiv c_{25}^{(k)} z^2 + d_{25}^{(k)} z^3 \tag{5}$$

Note that Eqs. (3) constrain all functions to pass through the origin at $z = 0$, whereas the new scheme frees the shear warping functions $g_{15}^{(k)}$ and $g_{24}^{(k)}$ to have an offset from the origin when they do not contain the reference surface. Also note that the quadratic term has been dropped and that the form of the shear coupling functions $g_{14}^{(k)}$ and $g_{25}^{(k)}$ is unchanged from Eqs. (3). The rationale for eliminating the quadratic term in the shear-warping functions is that, for isotropic plates, elasticity solutions give rise to no linear terms in the shear strains (only quadratic ones).[1]

Applying the functions of Eqs. (5) to our [0/90/0] laminate also yields a 6×6 system of equations. The determinant of the resulting coefficient matrix, calling it $[\tilde{A}]$, is $\det[\tilde{A}] = 108\hat{z}^3$, where $\hat{z}$ is the ratio of the thickness of the middle ply to the laminate thickness. This well-behaved determinant causes trouble only when $\hat{z}$ approaches zero (i.e., the middle ply vanishes). Moreover, this determinant is independent of the material properties. But the shear coupling functions are still constrained to pass through the origin. Geometrically nonlinear finite element analysis of thick shell problems[4] revealed that this restriction caused the ill-behaved warping functions to persist. Therefore, these coupling functions were also modified, leading to

$$g_{14}^{(k)} \equiv a_{14}^{(k)} + d_{14}^{(k)} z^3, \qquad g_{15}^{(k)} \equiv a_{15}^{(k)} + z + d_{15}^{(k)} z^3$$
$$g_{24}^{(k)} \equiv a_{24}^{(k)} + z + d_{24}^{(k)} z^3, \qquad g_{25}^{(k)} \equiv a_{25}^{(k)} + d_{25}^{(k)} z^3 \tag{6}$$

for plies not containing the reference surface.

III. Results

A total Lagrangian corotational finite element technique[2,5] is used to assess the modified warping functions against the elasticity solutions of Pagano[3] for the infinitely wide plate strip loaded with a transverse sinusoidal pressure load. The load $p(x)$ and the normalized center displacement $\bar{w}$ are defined as

$$p(x) = p_0 \sin\left[\pi \frac{(x + L/2)}{L}\right], \qquad \bar{w} = \frac{100\, E_{22} h^3 w_c}{p_0 L^4} \tag{7}$$

where w_c is the displacement at the center of the strip and L is the strip length.

The material properties used in the analysis are

$$E_{11} = 172.37 \text{ GPa}, \qquad E_{22} = E_{33} = 6.8948 \text{ GPa}$$
$$\nu_{12} = \nu_{13} = \nu_{23} = 0.25 \tag{8a}$$
$$G_{12} = G_{13} = 3.4474 \text{ GPa}, \qquad G_{23} = 1.37896 \text{ GPa} \tag{8b}$$

These shear moduli result in a value for $\tilde{G}$ of $\frac{5}{2}$. A length dimension of $L = 400$ mm was held constant, and the thickness h was varied to generate different values of $S = L/h$. To model half the strip, 20 elements were used, and so the element width was 10 mm. The boundary conditions used were as follows at $x = 0$ (yz-symmetric):

$$u = u_{,y} = v = v_{,x} = v_{,y} = w_{,x} = w_{,y} = \gamma_4 = \gamma_5 = 0 \tag{9a}$$

at $x = L/2$ (simple):

$$u_{,y} = v = v_{,x} = v_{,y} = w = w_{,y} = \gamma_4 = 0 \tag{9b}$$

and along $y = 0$ and $y = 2b$ (plane strain and symmetry):

$$u_{,y} = v = v_{,x} = v_{,y} = w_{,y} = \gamma_4 = 0 \tag{9c}$$

The middle of the middle ply is used as the reference surface.

In Figs. 1a and 1b, the behaviors of the shear warping function for the two methods presented herein are shown for the case of $\tilde{G} = \frac{5}{2}$. In each of the two plots, nine curves, representing the displacement function g_{15} for $\hat{z} = 0.1, 0.2, \ldots, 0.9$, are shown. In Fig. 1a, the g_{15} of Eqs. (3) are used to represent the function in every ply. This leads to the erratic behavior of the displacement function as shown. Note that the poorly behaved warping function can give artificially large (compare Fig. 1b) displacements, as well as displacements in the wrong direction.

In Fig. 1b, the improved method as suggested by Pai[1] is shown, wherein Eqs. (3) are used for the middle ply (which contains the reference surface) and Eqs. (6) are used for the outer plies. The g_{15} generated by this technique is well behaved; note that the g_{15} of Fig. 1a approaches that of Fig. 1b only as $\hat{z}$ approaches 1, i.e., the laminate tends toward a single ply. (The kinks in the curves represent the interlaminar boundaries.)

Displacements and stresses generated by the program approximate the elasticity solution to varying degrees of success. The displacement $\bar{u}_1 = (E_{22} u_1)/(p_0 h)$ is matched very well except in the middle (90-deg) ply (Fig. 1c). The reason for this difference between the $\bar{u}_1$ displacement function of the current work and that of Pagano's elasticity solution is that the slope of the displacement $u_1^{(2)}$ is constrained to take on the value of the shear γ_5, at the reference surface. Results are compared to those based upon classical laminated plate theory in Fig. 1c.

The in-plane stress $\bar{J}_{11} = J_{11}/p_0$ at the plate strip center is plotted in Fig. 1d, and the transverse shear stress $\bar{J}_{13} = J_{13}/p_0$ is shown in Fig. 1e. The current results model the stresses fairly well in the outer plies but only in the average sense over the center ply.

IV. Conclusions

Although many possible combinations of incomplete cubics are possible for representing the transverse displacements in a composite laminate, only a few make physical sense. Among these, a particular choice was shown to be susceptible to irregularities and singularities, in that an unfortunate choice of ply thicknesses and shear moduli lead to laminates overly stiff in shear. Modifications to the displacement functions were shown to eliminate this behavior and yield a transverse shear stress distribution having some of the features associated with the elasticity solution, namely, continuous shear stresses and in-plane displacements throughout the laminate thickness and the discontinuous transverse shear strains.

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